

Ising model on the edge-dual of random networks

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We consider the Ising model on the edge dual of uncorrelated random networks with arbitrary degree distribution. These networks have a finite clustering in the thermodynamic limit. High- and low-temperature expansions of Ising model on the edge dual of random networks are derived. A detailed comparison of the critical behavior of Ising model on scale free random networks and their edge dual is presented.

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I. INTRODUCTION

It is evident that a variety of natural and artificial systems can be described in terms of complex networks, in which the nodes represent typical units and the edges represent interactions between pairs of units [1–3]. Clearly, identifying structural and universal features of these networks is the first step in understanding the behavior of these systems [4–9]. Intensive research in recent years has revealed peculiar properties of complex networks which were unexpected in the conventional graph theory [10]. Among these one can refer to the scale free behavior of degree distribution [5] $P(k)$, where degree denotes the number of nearest neighbors of a node. From another point of view one is interested in the effect of structural properties of complex network on the collective behavior of systems living on these networks [11–18]. Percolation and Ising models (or in general the Potts model) are typical examples of statistical mechanics which have intensively been studied on uncorrelated random networks with given degree distributions [19–29]. By uncorrelated random network we mean those in which the degree of two neighbors are independent random variables. These networks are identified only by a degree distribution $P(k)$ and have the maximum possible entropy. The locally treelike nature of these networks provides a good condition to apply the recurrence relations to study the collective behavior of interesting models [22,29,30]. It is seen that, depending on the level at which the higher moments of $P(k)$ become infinite, one encounters different critical behaviors that could be derived from a Landau-Ginzburg theory [19]. This in turn reflects the mean field nature of these behaviors.

In this paper we are going to study the Ising model on the edge dual of uncorrelated random networks with a given degree distribution. These kinds of networks have already been introduced in the context of graph theory [10]. Given a network G , its edge dual \tilde{G} can be constructed as follows, see also Fig. 1: one puts a node in place of each edge of G and connects each pair of these new nodes if they are emanating from the same node of G . Such networks have been useful among other things for the study of the maximum matching

problem [31] as well as topological phase transitions in random networks [32].

The interesting point about the edge-dual networks is that they generally have a large degree of clustering even when the underlying network is treelike. Moreover, as was shown in Ref. [33], the edge dual of an uncorrelated random network is a positively correlated (or assortative) network [8], in which the nodes with the same degree have a higher tendency for being connected to each other. On the other hand, a high level of clustering along with a positive correlation between the degrees of neighboring nodes are the prominent features of social networks which differentiate them from the other kinds of real networks [34]. In addition one can easily show that the edge dual of an uncorrelated scale free network with $P(k) \propto k^{-\gamma}$ will be a network whose degree distribution

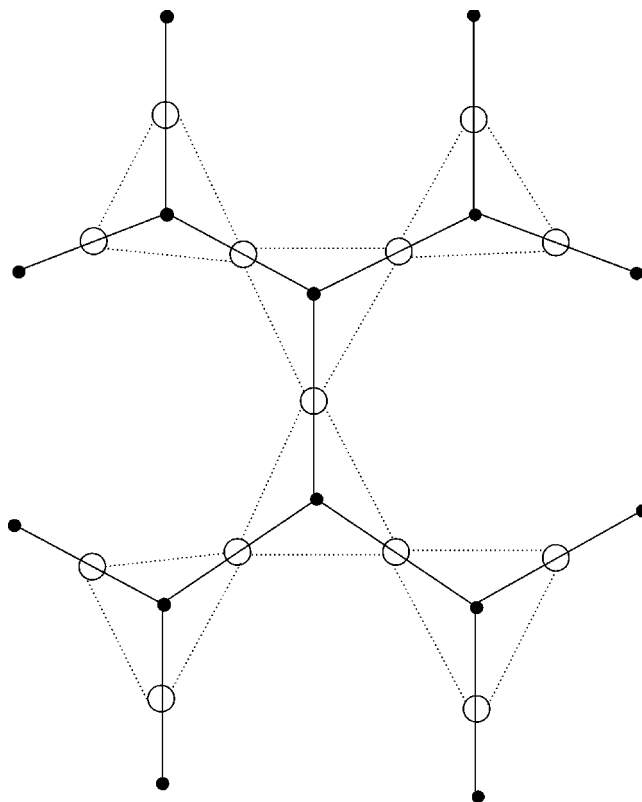


FIG. 1. Some part of the edge dual of a Bethe lattice with $z=3$. Empty circles and dotted lines denote the nodes and the edges of the edge-dual network, respectively.

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TABLE I. Comparison of the critical behavior of Ising model on scale free random networks [22] (written inside parenthesis) and its edge-dual network.

	Magnetization		Specific heat	Susceptibility
$\gamma > 5$	$\bar{\tau}^{1/2}(\tau^{1/2})$	$\tilde{H}^{1/3}$	cons.(cons.)	$\bar{\tau}^{-1}(\tau^{-1})$
$\gamma = 5$	$\bar{\tau}^{1/2}((\tau/\ln(\tau))^{1/2})$	$\tilde{H}/\ln(\tilde{H})^{1/3}$	$\ln(\bar{\tau})(1/\ln(\tau))$	$1/\bar{\tau}\ln(\bar{\tau})(\tau^{-1})$
$3 < \gamma < 5$	$\bar{\tau}^{1/2}(\tau^{1/(\gamma-3)})$	$\tilde{H}^{(8-\gamma)/9}$	$\bar{\tau}^{(\gamma-5)/2}(\tau^{(5-\gamma)/(\gamma-3)})$	$\bar{\tau}^{(3-\gamma)/2}(\tau^{-1})$

behaves as $\tilde{P}(k) \propto k^{-(\gamma-1)}$ for large degrees [33]. Summing up these points one finds that the edge dual of uncorrelated scale free networks would be a better candidate to model the structure of social networks. On the other hand, due to the presence of clustering and correlation in such networks, direct study of structural properties or of physical models defined on them is usually very difficult. However, one can use the above duality to adopt the techniques used in the context of random networks (e.g., the generating function formalism [20,21]) and obtain interesting results for the edge dual of uncorrelated random networks [33]. In this way one would be able to study the effects of clustering and correlation, for example, in the critical behavior of Ising model on such a network. So far we have a nearly complete picture of the critical behavior of this model on uncorrelated random networks, but there is still a lot which we need to know about the critical behavior of clustered and correlated networks.

Our basic result is depicted in Table I, where we have compared the critical behavior of Ising model on scale free networks and their edge dual networks. On the way to this basic result we have also studied as a preliminary step the Ising model on the edge dual of Bethe lattices. We have also developed a systematic high- and low-temperature expansion for the Ising model on edge-dual networks for arbitrary degree distributions.

Moreover as a by-product we have also shown that there is a simple relation between the partition function of an Ising model on a tree in which each spin interacts with its nearest and next-nearest neighbors and the partition function of an Ising model on its edge dual with only nearest-neighbor interactions but in the presence of magnetic field.

The paper is organized as follows. In Sec. II we use recurrence method for the study of Ising model on the edge dual of Bethe lattices. The same method is applied in Sec. III to study the critical behavior of Ising model on the edge dual of scale free networks. In Sec. IV a relationship between the Ising model on a tree and on its edge dual is derived. High- and low-temperature expansions of the partition function of the Ising model on the edge dual of a random network are given in Sec. V. The conclusions are presented in Sec. VI.

II. ISING MODEL ON THE EDGE DUAL OF BETHE LATTICES

The Bethe Lattice [30] is defined as a regular tree network (with no loop) where all nodes have the same degree z . Let us consider Ising spins on the edges of this network and let them interact with an external magnetic field \tilde{h} and with each

other if their corresponding edges are incident on the same node of the Bethe lattice. We can write the average magnetization of a spin lying on an arbitrary edge using the following recurrence relation [30]

$$\bar{m} = \frac{e^{\tilde{H}}g_+(0) - e^{-\tilde{H}}g_-(0)}{e^{\tilde{H}}g_+(0) + e^{-\tilde{H}}g_-(0)}, \tag{1}$$

where $\tilde{H} := \tilde{h}/T$ and $g_+(0)$ and $g_-(0)$ are, respectively, the partition functions for the system of spins on one side of the central spin when it is up or down. That is,

$$g_S(0) = \sum_{\{S_R\}} e^{\tilde{K} \sum_{i \in S} s_i s_{i+1} + \tilde{K} \sum_{(ij)} s_i s_j + \tilde{H} \sum_i s_i}, \tag{2}$$

where, as before, $\tilde{K} := \tilde{J}/T$. Note that in this partition function only spins on one side of S (for instance, the right-hand side spins, denoted by $\{S_R\}$) appear. Similar to $g_S(0)$ one can define a partition function $g_S(l)$ which gives the partition function of the branch of the lattice which stems from a node at layer l where the value of its spin has been fixed to S . These partition functions can be related to each other recursively as follows, see also Fig. 2: on the right hand side of S there are $z-1$ spins which interact with S and with each other. We can write $g_S(l)$ as a sum over different configurations of these spins. For each configuration we will have a term proportional to $g_+^r(l+1)g_-^{z-1-r}(l+1)$, where r will be the number of up spins in such a configuration. Moreover we have to consider another factor which takes into account the Boltzmann factor associated to this configuration of spins.

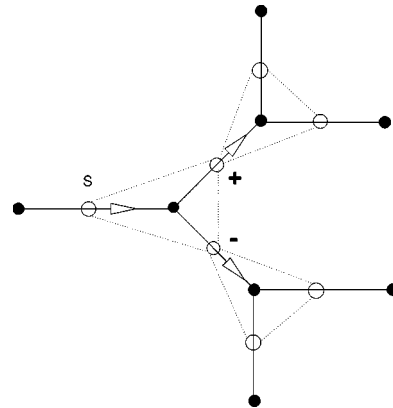


FIG. 2. The partition function of spins on the branch stems from S can recursively be written in terms of the partition functions of branches now stem from nearest neighbors of S .

Energy of a configuration in which r of these spins are up is the sum of three parts, a part given by their interaction with the external magnetic field equal to $-\tilde{h}(2r-z+1)$, a part from their interactions with spin S equal to $-\tilde{J}[Sr-S(z-1-r)]$ and finally a part given by interactions between themselves equal to $-\tilde{J}[r(r-1)/2+(z-1-r)(z-1-r-1)/2-r(z-1-r)]$. Summing up the above arguments we arrive at

$$g_S(l) = \sum_{r=0}^{z-1} \binom{z-1}{r} e^{\tilde{H}(2r-z+1)+\tilde{K}[S(2r-z+1)-2r(z-1-r)+(z-1)(z-2)/2]} \times g_+^r(l+1)g_-^{z-1-r}(l+1). \quad (3)$$

Returning to Eq. (1), magnetization of the central spin can be rewritten in a simpler form

$$\tilde{m} = \frac{e^{2\tilde{H}} - x_0^2}{e^{2\tilde{H}} + x_0^2} = \frac{e^{2\tilde{H}} - e^{-2y_0}}{e^{2\tilde{H}} + e^{-2y_0}}, \quad (4)$$

where we have defined $x_l := g_-(l)/g_+(l) =: e^{-y_l}$. When the magnetic field is positive we have $g_-(l) < g_+(l)$ thus y_l is a positive quantity which plays a role similar to the magnetic field and thus can be interpreted as the local field experienced by a spin at distance l from the central spin S . Now using Eq. (3), the recurrence relation for y_l reads

$$y_l = -\ln \left(\frac{\sum_r \binom{z-1}{r} e^{(2\tilde{H}+y_{l+1})r+\tilde{K}[z-1-2r-2r(z-1-r)]}}{\sum_r \binom{z-1}{r} e^{(2\tilde{H}+y_{l+1})r+\tilde{K}[2r-z+1-2r(z-1-r)]}} \right). \quad (5)$$

Setting $\tilde{H}=0$ and starting from distant ($l \gg 1$) spins with $y \ll 1$ one could obtain the values of y for deeper spins in a step by step manner using the above relation until one arrives at y_0 . Equation (4) tells us that we will have magnetization in this case only if y_0 is different from zero. It is evident that a stable nonzero solution for y_0 is possible only when the right-hand side of the recurrence relation for y_l (when plotted versus y_{l+1}) has a slope greater than or equal to 1. The equality will provide the critical temperature of the system which turns out to be given by

$$\frac{\sum_r r \binom{z-1}{r} e^{-\tilde{K}_c 2r(z-1-r)} 2 \sinh[\tilde{K}_c(2r-z+1)]}{\sum_r \binom{z-1}{r} e^{\tilde{K}_c[z-1-2r(z-r)]}} = 1. \quad (6)$$

Unfortunately it is not possible to derive a closed relation for \tilde{K}_c . In Fig. 3 we have computed this quantity numerically and compared it with the corresponding quantity in the Bethe lattice itself. In the latter case the critical temperature reads [30]

$$\tanh K_c = \frac{1}{z-1}. \quad (7)$$

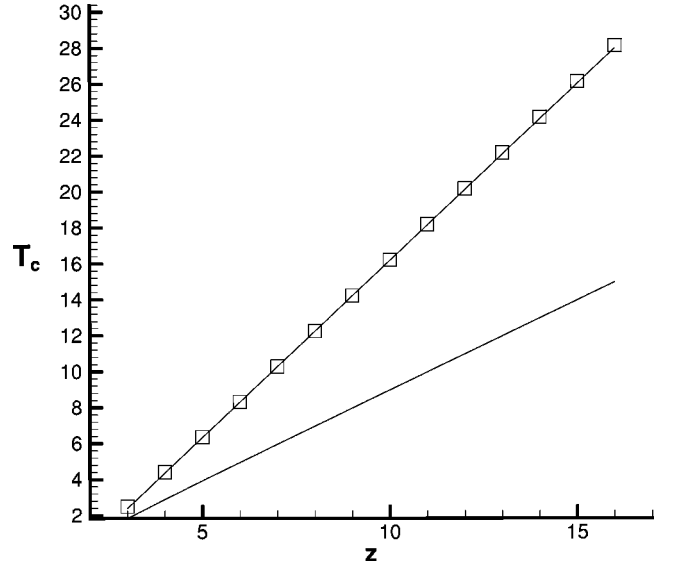


FIG. 3. T_c (in units of J) for the Ising model on Bethe lattices (lower curve) and their edge dual (upper curve).

As Fig. 3 shows \tilde{T}_c is much greater than T_c which is as expected due to the larger number of interactions in the edge-dual network. In the figure we also show a linear fit $\tilde{T}_c = a_0 + a_1 z$ to the numerical data for \tilde{T}_c with $a_0 = -3.53 \pm 0.02$ and $a_1 = 1.97 \pm 0.002$. As long as the critical behavior of the system is concerned we expect to see a standard mean field behavior as in the usual Ising model in spatial dimensions greater than $d_c = 4$. We will further discuss these issues in the next section.

III. ISING MODEL ON THE EDGE DUAL OF RANDOM NETWORKS

In this section we generalize the results of the previous section to the case of Ising model on the edge dual of random networks with a given degree distribution $P(k)$. In this case a spin on the edge of such a random network will encounter $k_1 - 1$ and $k_2 - 1$ nearest neighbors on its right- and left-hand sides, respectively. These numbers are random variables given by the degree distribution of nearest neighbors in the random network $\Pi(k)$.

A. General arguments

Along the lines of Sec. II we can write the magnetization of a spin on an edge of random network with end point nodes of degrees k_1 and k_2 as

$$\tilde{m}_{k_1 k_2} = \frac{e^{2\tilde{H}} - e^{-y_0(k_1) - y_0(k_2)}}{e^{2\tilde{H}} + e^{-y_0(k_1) - y_0(k_2)}}, \quad (8)$$

where now y 's are random variables depending on distance and degree of end point nodes. Obviously magnetization of an arbitrary spin is given by

$$\tilde{m} = \sum_{k_1, k_2} \Pi(k_1)\Pi(k_2) \frac{e^{2\tilde{H}} - e^{-\gamma_0(k_1)-\gamma_0(k_2)}}{e^{2\tilde{H}} + e^{-\gamma_0(k_1)-\gamma_0(k_2)}}. \quad (9)$$

As before we used the notation $x_l(k) := e^{-\gamma_l(k)}$ and $x_l(k) := g_-(l; k) / g_+(l; k)$, where $g_S(l; k)$ is the partition function for the cluster beyond the spin S at distance l from the central spin. As in the case of Bethe lattices (Fig. 2) these quantities can be related to $g_{S'}(l+1; k')$'s by the following relations:

$$g_S(l; k) = \sum_{r=0}^{k-1} e^{\tilde{H}(2r-k+1) + \tilde{K}[S(2r-k+1) - 2r(k-1-r) + (k-1)(k-2)/2]} \times \sum_{r/k-1}^r \prod_{a=1}^r g_+(l+1; k_a) \prod_{b=r+1}^{k-1} g_-(l+1; k_b), \quad (10)$$

where the second sum is over different selections of r distinct spins from the set of $k-1$ neighboring spins after assigning indices 1 to r to them. Thus the relation for $x_l(k)$ has the form

$$x_l(k) = \frac{\sum_{r=0}^{k-1} e^{\tilde{H}2r + \tilde{K}[k-1-2r-2r(k-1-r)]} \sum_{k-1-r/k-1} \prod_{a=1}^{k-1-r} x_{l+1}(k_a)}{\sum_{r=0}^{k-1} e^{\tilde{H}2r + \tilde{K}[2r-k+1-2r(k-1-r)]} \sum_{k-1-r/k-1} \prod_{a=1}^{k-1-r} x_{l+1}(k_a)} \quad (11)$$

or in terms of y 's

$$y_l(k) = -\ln \left(\frac{\sum_{r=0}^{k-1} e^{\tilde{H}2r + \tilde{K}[k-1-2r-2r(k-1-r)]} \sum_{k-1-r/k-1} e^{-\sum_{a=1}^{k-1-r} y_{l+1}(k_a)}}{\sum_{r=0}^{k-1} e^{\tilde{H}2r + \tilde{K}[2r-k+1-2r(k-1-r)]} \sum_{k-1-r/k-1} e^{-\sum_{a=1}^{k-1-r} y_{l+1}(k_a)}} \right). \quad (12)$$

Let us also derive a relation for the average energy of Ising model on the edge dual of random networks in the absence of magnetic field. First note that we can write this quantity as a sum over the interaction energy associated to the spins on the edges emanating from the same node of G , that is

$$\tilde{E} = \sum_i e_i = \sum_i \left(\sum_{\langle (ij)(ik) \rangle} -\tilde{J}S_{(ij)}S_{(ik)} \right). \quad (13)$$

Thus the thermodynamic average of the above quantity reads

$$\langle \tilde{E} \rangle = N \sum_k P(k) \langle e_k \rangle, \quad (14)$$

where $\langle e_k \rangle$ is the average energy associated to a node of degree k . We are able to write this quantity in terms of g 's by summing over different configurations of spins on the edges of such a node. As before we sum over configurations in which r spins out of these k spins are up. For each such configuration we include an appropriate Boltzmann weight as before. Denoting the interaction energy of these spins by

$$e_k := -\tilde{J} \left(\frac{r(r-1)}{2} + \frac{(k-r)(k-r-1)}{2} - r(k-r) \right)$$

we obtain

$$\langle e_k \rangle = \frac{\sum_{r=0}^k e_k e^{\tilde{K}[r(r-1)/2 + (k-r)(k-r-1)/2 - r(k-r)]} \sum_{r/k} \prod_{a=1}^r g_+(0; k_a) \prod_{b=r+1}^k g_-(0; k_b)}{\sum_{r=0}^k e^{\tilde{K}[r(r-1)/2 + (k-r)(k-r-1)/2 - r(k-r)]} \sum_{r/k} \prod_{a=1}^r g_+(0; k_a) \prod_{b=r+1}^k g_-(0; k_b)}. \quad (15)$$

After some algebra this relation takes the following simpler form in terms of y 's

$$\langle e_k \rangle = -\tilde{J} \left(\frac{k(k-1)}{2} - \frac{\sum_{r=0}^k 2r(k-r) e^{-\tilde{K}2r(k-r)} \sum_{k-r/k} e^{-\sum_{a=1}^{k-r} y_0(k_a)}}{\sum_{r=0}^k e^{-\tilde{K}2r(k-r)} \sum_{k-r/k} e^{-\sum_{a=1}^{k-r} y_0(k_a)}} \right). \quad (16)$$

Equations (9), (12), and (16) are the exact relations for magnetization, effective fields y , and energy of the system.

B. The effective medium approximation

In this section we simplify the relations obtained in the previous subsection using the effective medium approximation [22,29] applied satisfactorily to the study of Ising model on uncorrelated random networks. It is believed that this approximation takes in a good way into account the effects of high degree nodes which play an essential role in determining the critical behavior of the system specifically in inhomogeneous network having scale free degree distribution.

To this end we rewrite the relations derived above as if y 's are independent of k , the degree of the end point nodes. This is achieved if we use the same g for all the spins which are at the same distance from the central spin. The only explicit dependence on k enters Eq. (12) which must be averaged over using the degree distribution of nearest neighbors $\Pi(k)$. We emphasize that this approximation is exact if we expand our relations for small y 's and keeping only the linear term. Note that we are finally interested in the critical behavior of the system where y 's tend to zero and thus we expect the above approximation to work well in the critical region. Consequently we use the following relations to extract the critical behavior of Ising Model on the edge dual of an uncorrelated random network:

$$\tilde{m} = \frac{e^{2\tilde{H}} - e^{-2y_0}}{e^{2\tilde{H}} + e^{-2y_0}} \quad (17)$$

for the magnetization,

$$y_l = - \sum_k \Pi(k) \ln \left(\frac{\sum_{r=0}^{k-1} \binom{k-1}{r} e^{(2\tilde{H}+y_{l+1})r + \tilde{K}[k-1-2r-2r(k-1-r)]}}{\sum_{r=0}^{k-1} \binom{k-1}{r} e^{(2\tilde{H}+y_{l+1})r + \tilde{K}[2r-k+1-2r(k-1-r)]}} \right), \quad (18)$$

for the recurrence relations defining y 's and

$$\langle \tilde{E} \rangle = -N\tilde{J} \sum_k P(k) \times \left(\frac{k(k-1)}{2} - \frac{\sum_{r=0}^k \binom{k}{r} 2r(k-r) e^{-\tilde{K}2r(k-r) + y_0 r}}{\sum_{r=0}^k e^{-\tilde{K}2r(k-r) + y_0 r}} \right) \quad (19)$$

for the average of energy in the absence of magnetic field.

At this stage it is instructive to note that we can obtain the correlation between the central spin and a spin at distance l by taking derivative of \tilde{m} with respect to \tilde{H}_l , the magnetic field acting on such a spin. To this end we need also to label the magnetic fields along with the y 's in the above relations. Indeed in Eqs. (17) and (18) the magnetic field has a similar index to that of y . On the other hand, we have

$$\tilde{\chi}_l = \left. \frac{\partial \tilde{m}}{\partial \tilde{H}_l} \right|_{\{\tilde{H}_i=0\}} = \tilde{n}_l \tilde{G}_c(0, l), \quad (20)$$

where $\tilde{n}_l = 2[(\langle k^2 \rangle - \langle k \rangle) / \langle k \rangle]^l$ is the number of spins at distance l from the central spin in the edge dual of random network and $\tilde{G}_c(0, l) = \langle SS_l \rangle - \langle S \rangle \langle S_l \rangle$. Note that susceptibility is given by $\tilde{\chi} = \sum_l \tilde{\chi}_l$ where, from Eqs. (20) and (17), $\tilde{\chi}_l$ reads

$$\tilde{\chi}_l = \left. \frac{\partial \tilde{m}}{\partial y_0} \prod_{i=0}^{l-2} \left(\frac{\partial y_i}{\partial y_{i+1}} \right) \frac{\partial y_{l-1}}{\partial \tilde{H}_l} \right|_{\{\tilde{H}_i=0\}}. \quad (21)$$

If spins are deep enough in the network we can take all the y 's equal to each other, so for their derivatives. After making this approximation we obtain

$$\tilde{\chi}_l = \left. \frac{\partial \tilde{m}}{\partial y_0} \left(\frac{\partial y_i}{\partial y_{i+1}} \right)^{-1} \frac{\partial y_{l-1}}{\partial \tilde{H}_l} \right|_{\{\tilde{H}_i=0\}} e^{-\frac{l}{\tilde{\lambda}}}, \quad (22)$$

where

$$\tilde{\lambda} = - \frac{1}{\ln \left(\frac{\partial y_i}{\partial y_{i+1}} \right)}. \quad (23)$$

Here the index i is only to distinguish between y 's in two subsequent shells and we will eventually set all the y 's equal to each other. This quantity is determined from the fixed point of Eq. (18). Consequently the length scale $\tilde{\lambda}$ is determined from Eq. (23) and as expected, it will become infinite in the critical point, that is when $\partial y_i / \partial y_{i+1} = 1$. It is this critical behavior that gives rise to the critical behavior of $\tilde{\chi}$. Note, however, that $\tilde{\lambda}$ is not the correlation length which is determined from the long distance behavior of $\tilde{G}_c(0, l)$.

C. The critical behavior

Using Eq. (18) it is not difficult to see that y_0 is nonzero only for temperatures less than \tilde{T}_c which satisfies

$$\sum_k \Pi(k) \frac{\sum_{r=0}^{k-1} r \binom{k-1}{r} e^{-\tilde{K}_c 2r(k-1-r)} 2 \sinh[\tilde{K}_c(2r-k+1)]}{\sum_r \binom{k-1}{r} e^{\tilde{K}_c(k-1-2r(k-r))}} = 1. \tag{24}$$

It is a simple generalization of Eq. (6). This equation tells us that if $\tilde{K}_c \rightarrow 0$, i.e., at high temperatures, we have $\tilde{K}_c \sim \langle k \rangle / (\langle k^2 \rangle - \langle k \rangle)$. In other words the critical temperature of the system becomes infinite only when the second moment of $P(k)$ is infinite. This is the same behavior observed in uncorrelated random networks [22,23]. Thus even the finite value of clustering of edge dual networks cannot significantly alter the critical point although $\tilde{T}_c \gg T_c$ as we saw in the case of Bethe lattices in Sec. IV.

Now let us limit ourselves to the critical region where y , \tilde{H} , and $\tilde{\tau} := |(T - \tilde{T}_c) / \tilde{T}_c|$ are very small. We want an expansion of \tilde{m} , y , and $\langle \tilde{E} \rangle$ in terms of small deviations from the critical point. First note that if we change the sign of \tilde{H} , then by definition the sign of y changes too and thus the magnetization is reversed. On the other hand, energy does not change under this change of sign. Any expansion of these quantities in terms of \tilde{H} and y must satisfy these symmetries. We summarize these arguments in the following expansions:

$$\begin{aligned} \tilde{m} &\approx \tilde{H} + y, \\ y &\approx a_1(2\tilde{H} + y) + a_3(2\tilde{H} + y)^3, \\ \langle \tilde{E} \rangle &\approx b_0 + b_2 y^2, \end{aligned} \tag{25}$$

where the coefficients a_1, a_3, b_0 , and b_2 are found to be

$$\begin{aligned} a_1 &= 1 + O(\langle k^4 \rangle) \tilde{\tau}, \quad a_3 \sim O(\langle k^4 \rangle), \\ b_0 &\sim O(\langle k^2 \rangle), \quad b_2 \sim O(\langle k^4 \rangle). \end{aligned} \tag{26}$$

In this equation the averages are taken with respect to the degree distribution of random network $P(k)$. To be more specific let us consider a definite degree distribution, that is the well known scale free distribution $P(k) \propto k^{-\gamma}$. We consider several cases depending on the value of γ .

1. The case $\gamma > 5$

In this case the edge dual network behaves as a scale free network [33], with $\tilde{\gamma} = \gamma - 1 > 4$. Moreover, all the coefficients appearing in the expansions of Eq. (25) are finite. It is easy to show using Eq. (25) and (26) that y is given by the following relations:

$$\begin{aligned} y &\sim \tilde{\tau}^{1/2}, \quad \tilde{H} = 0, \\ y &\sim \tilde{H}^{1/3}, \quad \tilde{\tau} = 0, \\ y &\sim \frac{\tilde{H}}{\tilde{\tau}}, \quad \tilde{H} \neq 0, \tilde{\tau} \neq 0. \end{aligned} \tag{27}$$

The critical behavior of the other quantities can easily be derived from these relations

$$\begin{aligned} \tilde{m} &\sim \tilde{\tau}^{1/2}, \quad \delta\tilde{C} \sim \text{const.}, \quad \tilde{\chi} \sim \tilde{\tau}^{-1}, \quad \tilde{H} = 0, \\ \tilde{m} &\sim \tilde{H}^{1/3}, \quad \tilde{\tau} = 0, \end{aligned} \tag{28}$$

where $\delta\tilde{C}$ is the change of specific heat through the critical point. Here we have only shown dependence of interesting quantities on $\tilde{\tau}$ and \tilde{H} . Clearly these behaviors are those of the standard mean field model seen in the Ising model in spatial dimensions greater than $d_c = 4$. This behavior is also seen in the case of Ising model on uncorrelated scale free random networks with $\gamma > 5$ [22,23]. Note that due to the finiteness of all the moments of degree distribution in Bethe lattices, the critical behavior of Ising model on their edge dual network also lies in this class.

2. The case $\gamma = 5$

Now $\tilde{\gamma} = 4$. Some of the coefficients in expansions of Eq. (25) become infinite. To avoid these divergences which are an artifact of our expansion, we set a cut off for degrees which is proportional to $1/y$. Indeed in our expansion we used the fact that $ky \ll 1$ where k is the degree of a nearest neighbor. Fortunately we are interested in the critical behavior where $y \rightarrow 0$ and the the above arguments work well in that region [22].

Considering the above arguments, we find

$$\begin{aligned} y &\sim \tilde{\tau}^{1/2}, \quad \tilde{H} = 0, \\ y &\sim \left(\frac{\tilde{H}}{\ln(\tilde{H})} \right)^{1/3}, \quad \tilde{\tau} = 0, \\ y &\sim \frac{\tilde{H}}{\tilde{\tau} \ln(\tilde{\tau})}, \quad \tilde{H} \neq 0, \tilde{\tau} \neq 0. \end{aligned} \tag{29}$$

Thus the interesting quantities behave as

$$\begin{aligned} \tilde{m} &\sim \tilde{\tau}^{1/2}, \quad \delta\tilde{C} \sim \ln(\tilde{\tau}), \quad \tilde{\chi} \sim \frac{1}{\tilde{\tau} \ln(\tilde{\tau})}, \quad \tilde{H} = 0, \\ \tilde{m} &\sim \left(\frac{\tilde{H}}{\ln(\tilde{H})} \right)^{1/3}, \quad \tilde{\tau} = 0. \end{aligned} \tag{30}$$

3. The case $3 < \gamma < 5$

In this case the degree distribution of edge dual network will have the exponent $2 < \tilde{\gamma} < 4$. Again we have to take into

account the divergences appearing in the expansion coefficients. From Eqs. (25) and (26) we find

$$\begin{aligned} y &\sim \tilde{\tau}^{1/2}, \quad \tilde{H} = 0, \\ y &\sim \tilde{H}^{(8-\gamma)/9}, \quad \tilde{\tau} = 0, \\ y &\sim \frac{\tilde{H}}{\tilde{\tau}^{(\gamma-3)/2}}, \quad \tilde{H} \neq 0, \quad \tilde{\tau} \neq 0. \end{aligned} \quad (31)$$

For the thermodynamic quantities we find

$$\begin{aligned} \tilde{m} &\sim \tilde{\tau}^{1/2}, \quad \delta\tilde{C} \sim \tilde{\tau}^{(\gamma-5)/2}, \quad \tilde{\chi} \sim \tilde{\tau}^{-(\gamma-3)/2}, \quad \tilde{H} = 0, \\ \tilde{m} &\sim \tilde{H}^{(8-\gamma)/9}, \quad \tilde{\tau} = 0. \end{aligned} \quad (32)$$

We do not consider the case $\gamma \leq 3$ since in this region $\tilde{\gamma} \leq 2$ and the average number of neighbors is infinite in the edge dual network although it is still finite in the corresponding random network.

The above results show that these critical behaviors are very different from the ones seen in the uncorrelated scale free networks [22,23], see Table I. For example, here the magnetization always behaves similar to the standard mean field case $\tilde{m} \sim \tilde{\tau}^{1/2}$, but in uncorrelated scale free networks this behavior is only seen for $\gamma > 5$ where all quantities are of the standard mean field type [22,23].

IV. A RELATION BETWEEN ISING MODEL ON A TREE AND ITS EDGE-DUAL NETWORK

Let us consider an Ising model (with values of spin taking only ± 1) on a tree graph G with nearest- and next-nearest neighbor interactions of strength J_1 and J_2 in the absence of magnetic field. The Hamiltonian is

$$E = -J_1 \sum_{\langle ij \rangle_1} S_i S_j - J_2 \sum_{\langle ij \rangle_2} S_i S_j, \quad (33)$$

where $\langle ij \rangle_1$ and $\langle ij \rangle_2$ denote the nearest- and next-nearest neighbors, respectively. For any given configuration of spins we can assign a unique configuration of spin variables (again taking values ± 1) to the edges of the graph: any edge which connects two nodes having the variables S_i and S_j is assigned a value $S_{(ij)} := S_i S_j$, see Fig. 4. On the other hand, for any configurations of spins on the edges $S_{(ij)}$, there are two possible configurations of spins on the nodes, which are obtained by flipping all the spins S_i on the graph. Therefore there is a two to one correspondence between the spin configurations on the nodes of the graph and the edges of the graph.

Now if we write the above Hamiltonian in terms of spins of edges we get

$$E = -J_1 \sum_{\langle ij \rangle_1} S_{(ij)} - J_2 \sum_{\langle ij \rangle_2} S_{(ik)} S_{(kj)}, \quad (34)$$

where k is the common nearest neighbor of nodes i and j . But this is the Hamiltonian of an Ising model on \tilde{G} , the edge

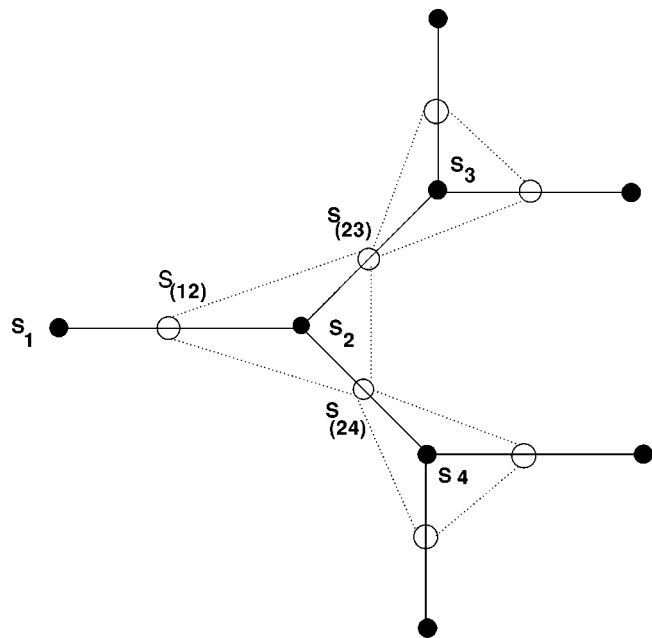


FIG. 4. To each edge (ij) we can assign a spin which is defined as the product of spins at the end point nodes of that edge, i.e., $S_{(ij)} := S_i S_j$.

dual of the tree, with nearest-neighbor interactions of strength $\tilde{J}_1 = J_2$ in presence of a magnetic field of magnitude $\tilde{h} = J_1$:

$$\tilde{E} = -\tilde{h} \sum_{\tilde{i}} S_{\tilde{i}} - \tilde{J}_1 \sum_{\langle \tilde{i}\tilde{j} \rangle_1} S_{\tilde{i}} S_{\tilde{j}}, \quad (35)$$

where \tilde{i} and \tilde{j} now denote the nodes of the edge dual graph. Taking into account the relation between configurations mentioned above we obtain

$$Z(J_1, J_2, h = 0, T) = 2\tilde{Z}(\tilde{J}_1 = J_2, \tilde{J}_2 = 0, \tilde{h} = J_1, T), \quad (36)$$

in which Z and \tilde{Z} are, respectively, the partition functions of the Ising model on G and \tilde{G} and T denotes the temperature. We should stress that the above relation is true only for tree graphs, since the presence of loops in G puts constraints on the values of spins which are assigned to the nodes of \tilde{G} . That is for any loop in G , the product of spins on its edges (equivalently the nodes of \tilde{G}) should be $+1$. Taking into account all these constraints makes the calculation of the partition function very difficult in the general case.

In the thermodynamic limit the two models have the same free energy density, that is, $f = \tilde{f}$, where $f := -T \ln Z/N$ and $\tilde{f} := -T \ln \tilde{Z}/\tilde{N}$ are, respectively, the free energy of the Ising model on tree and its edge dual. Here we have set the Boltzmann constant equal to one. Note that for a tree graph, $\tilde{N} = N - 1$. Moreover, it is easy to see that the average magnetization of a spin in \tilde{G}

$$\tilde{m} = -\frac{\partial \tilde{f}}{\partial \tilde{h}} \quad (37)$$

is equal to the average correlation of two neighboring spins in G

$$\langle S_i S_j \rangle = \frac{-\partial f}{\partial J_1}. \quad (38)$$

Knowing the average magnetization of spins in \tilde{G} as a function of magnetic field, we can write the free energy density from Eq. (37) as follows [30]:

$$\tilde{f}(\tilde{J}_1, \tilde{h}, T) = \int_{\tilde{h}}^{\infty} [\tilde{m}(h') - 1] dh' - \tilde{J}_1 - \tilde{h}. \quad (39)$$

Differentiation of the right-hand side with respect to \tilde{h} correctly gives the magnetization \tilde{m} and the integration constant $-\tilde{J}_1 - \tilde{h}$ can be understood from the fact that for very large magnetic fields when the first integral vanishes, the partition function is dominated by the configurations where all the spins are up, and hence the free energy per site is given by $-\tilde{J}_1 - \tilde{h}$.

Obviously if we replace \tilde{J}_1 with J_2 and \tilde{h} with J_1 we obtain f , the free energy per site of Ising model on G with nearest- and next-nearest-neighbor interactions. In this way any nonanalytic behavior of \tilde{f} will appear in f too.

V. HIGH AND LOW TEMPERATURE EXPANSION OF ISING MODEL ON EDGE-DUAL NETWORKS

Consider a random network which we denote by G . We assume that the probability of each node having degree k is equal to $P(k)$. Furthermore we assume that there is no correlations between the degrees of adjacent neighbors. The average number of nearest neighbors will be denoted by $n_1 := \langle k \rangle = \sum_k k P(k)$. The degree distribution of nearest neighbors [21] is given by $\Pi(k) = k P(k) / \langle k \rangle$. Thus the average number of next-nearest neighbors will be $n_2 := \langle k \rangle \sum_k (k-1) \Pi(k) = \langle k^2 \rangle - \langle k \rangle$.

The edge dual of such a network is denoted by \tilde{G} . Correspondingly every quantity pertaining to the dual network \tilde{G} will be designated by a tilde. In the following we will study the Ising model with only nearest-neighbor interactions on \tilde{G} . The partition function is

$$\tilde{Z} = \sum_{\{S_{\tilde{j}}\}} e^{\tilde{K} \sum_{\langle \tilde{i}\tilde{j} \rangle} S_{\tilde{i}} S_{\tilde{j}}}, \quad (40)$$

where the sum in the exponential is over the nearest neighbors on the edge dual of G . Here we have used the notation $\tilde{K} := \tilde{J}/T$.

A. High-temperature expansion

The above partition function can be written in a form appropriate for a high-temperature expansion [30]. To this end we write the exponential in the form

$$e^{\tilde{K} \sum_{\langle \tilde{i}\tilde{j} \rangle} S_{\tilde{i}} S_{\tilde{j}}} = \cosh^{\tilde{L}(\tilde{K})} \prod_{\langle \tilde{i}\tilde{j} \rangle} [1 + S_{\tilde{i}} S_{\tilde{j}} \tanh(\tilde{K})], \quad (41)$$

where $\tilde{L} := N n_2 / 2$ is the number of edges in the \tilde{G} . Inserting this in Eq. (40) and expanding the product we get a series of terms each corresponding to a subgraph of \tilde{G} . Summing over spin configurations only terms which represent even subgraphs (in them each node has an even number of edges) will survive [30] and we arrive at the following expression for the partition function:

$$\tilde{Z} = \cosh^{\tilde{L}(\tilde{K})} 2^{\tilde{N}} \sum_{\text{even subgraphs } g_e} \tilde{\eta}^{\tilde{L}(g_e)}, \quad (42)$$

where $\tilde{\eta} = \tanh(\tilde{K})$, $\tilde{N} = N n_1 / 2$ is the number of nodes of \tilde{G} , and $\tilde{L}(g_e)$ is the perimeter (the number of edges) of the even subgraph g_e .

Clearly at high temperatures the first and the second terms corresponding to triangles and squares need be kept in the expansion:

$$\tilde{Z} \approx \cosh^{\tilde{L}(\tilde{K})} 2^{\tilde{N}} (\tilde{N}_{\Delta} \tilde{\eta}^3 + \tilde{N}_{\square} \tilde{\eta}^4 + \dots). \quad (43)$$

The number of triangles in \tilde{G} has two parts: first, each triangle of G appears as a triangle in \tilde{G} too. Secondly, by definition of the edge-dual network, every triple of edges emanating from the same node in G make a triangle in \tilde{G} . The number of the latter types of triangles is given by the number of distinct choices of three edges of a node, summed over the nodes of G . Thus

$$\tilde{N}_{\Delta} = N_{\Delta} + \sum_i \binom{k_i}{3}, \quad (44)$$

in which k_i is the degree of node i in G . Since in an uncorrelated random network the number of triangles is a finite quantity in the thermodynamic limit [35], we can neglect the first term compared with the second one which has an infinite contribution in this limit. The same argument is applicable to the case of squares so we can approximate these numbers by

$$\tilde{N}_{\Delta} \approx \sum_i \binom{k_i}{3} = \frac{N}{3!} \sum_k k(k-1)(k-2)P(k),$$

$$\tilde{N}_{\square} \approx \sum_i \binom{k_i}{4} = \frac{N}{4!} \sum_k k(k-1)(k-2)(k-3)P(k). \quad (45)$$

Note that these relations become exact in the case of tree structures even for finite N .

B. Low-temperature expansion

We now return to Eq. (40), the original relation for the partition function. Note that we can rewrite it as

$$\tilde{Z} = \sum_{\{S_{(ij)}\}} e^{\tilde{K}/2 \sum_i [(\sum_{j|i} S_{(ij)})^2 - k_i]}, \quad (46)$$

where the first sum in the exponential is over nodes of G and the second is over nearest neighbors of that node. Let us write \tilde{Z} in a simpler way

$$\tilde{Z} = e^{-\tilde{K}L} \sum_{\{S_{(ij)}\}} e^{\tilde{K}/2 \sum_i m_i^2}, \quad (47)$$

where L is the number of edges in G and $m_i := \sum_{j|i} S_{(ij)}$. Using the identity

$$e^{(\tilde{K}/2)m_i^2} = \frac{1}{\sqrt{\frac{2\pi}{\tilde{K}}}} \int dx_i e^{-(\tilde{K}/2)(x_i^2 + 2m_i x_i)}, \quad (48)$$

we find

$$\tilde{Z} = e^{-\tilde{K}L} \left(\frac{\tilde{K}}{2\pi} \right)^{N/2} \sum_{\{S_{(ij)}\}} \int D\mathbf{x} e^{-(\tilde{K}/2) \sum_i (x_i^2 + 2x_i m_i)}, \quad (49)$$

where $D\mathbf{x} = \prod_i dx_i$ and i runs over all the nodes of G . We can now perform the sum over the spin configurations in the integrand. To this end we note in view of the definition of m_i

$$\sum_{\{S_{(ij)}\}} e^{-\tilde{K} \sum_i x_i m_i} = \sum_{\{S_{(ij)}\}} e^{-\tilde{K} \sum_{\langle ij \rangle} (x_i + x_j) S_{(ij)}}, \quad (50)$$

where $\sum_{\langle ij \rangle}$ sums over all the links of G . The sum can be transformed to

$$\begin{aligned} \prod_{\langle ij \rangle} (e^{-\tilde{K}(x_i + x_j)} + e^{\tilde{K}(x_i + x_j)}) &= e^{\tilde{K} \sum_{\langle ij \rangle} (x_i + x_j)} \prod_{\langle ij \rangle} (1 + e^{-2\tilde{K}(x_i + x_j)}) \\ &= e^{\tilde{K} \sum_i k_i x_i} \prod_{\langle ij \rangle} (1 + e^{-2\tilde{K}(x_i + x_j)}). \end{aligned} \quad (51)$$

Putting all these together we find

$$\tilde{Z} = e^{-\tilde{K}L} \left(\frac{\tilde{K}}{2\pi} \right)^{N/2} \int D\mathbf{x} e^{-(\tilde{K}/2) \sum_i (x_i^2 - 2k_i x_i)} \prod_{\langle ij \rangle} (1 + e^{-2\tilde{K}(x_i + x_j)}). \quad (52)$$

The product $\prod_{\langle ij \rangle} (1 + e^{-2\tilde{K}(x_i + x_j)})$ can now be expanded as a series of terms each corresponding to a subgraph g of G .

For any node i of the graph G , a factor $e^{-2\tilde{K}z_i x_i}$ should be taken into account in which z_i is the degree of that node in the subgraph. If a node i does not belong to the subgraph, $z_i = 0$. Any subgraph determines uniquely the sequence of integers $\{z_i; i = 1, \dots, N\}$. Note that $z_i \leq k_i \forall i$. For each such sequence the integral can be easily calculated yielding

$$\tilde{Z} = e^{-\tilde{K}L + (\tilde{K}/2) \sum_i k_i^2} \sum_g e^{-2\tilde{K} \sum_i z_i (k_i - z_i)}. \quad (53)$$

It is the central result of this subsection which can be used for a low-temperature expansion of Ising model on \tilde{G} . This formula incidentally shows that each subgraph g and its complement (the graph obtained when one removes all the links of g from G) give the same contribution to the partition function.

At very low temperatures, $\tilde{K} \rightarrow \infty$, only the empty graph for which all $z_i = 0$ and its complement for which all $z_i = k_i$ contribute yielding

$$\begin{aligned} Z_0 &= 2e^{-\tilde{K}L + (\tilde{K}/2) \sum_i k_i^2} = 2e^{-\tilde{K}L + (\tilde{K}/2)N \sum_k k^2 P(k)} \\ &= 2e^{-\tilde{K}(N/2) \langle k \rangle + (\tilde{K}/2)N \langle k^2 \rangle} \end{aligned} \quad (54)$$

resulting in a free energy per site equal to

$$\tilde{f}_0 = -\frac{\tilde{J}}{2} (\langle k^2 \rangle - \langle k \rangle), \quad (55)$$

where we have used the relation $\tilde{K} = \tilde{J}/T$.

The next to leading order term comes from subgraphs which have only one link, (we multiply their contribution by 2 to account for their complements). This will give

$$\tilde{Z} = \tilde{Z}_0 \left(1 + \sum_{\langle ij \rangle} e^{-2\tilde{K}(k_i + k_j - 2)} + \dots \right), \quad (56)$$

where the sum is over all the links of G . This can be written as

$$\tilde{Z} = \tilde{Z}_0 \left(1 + N \frac{\langle k \rangle}{2} e^{4\tilde{K}} \ll e^{-2\tilde{K}(k+k')} \gg + \dots \right), \quad (57)$$

where $\ll \gg$ denotes the average with respect to the two point function $P(k, k')$, the probability of two nodes of degrees k and k' to be neighbors. For uncorrelated networks one has $P(k, k') = (2 - \delta_{k, k'}) \Pi(k) \Pi(k')$. This procedure can be followed for higher order contributions.

VI. CONCLUSION

In summary we studied the Ising model with nearest-neighbor interactions on the edge dual of uncorrelated random networks. As a simple example we studied the Ising model on the edge-dual of Bethe lattices using the well known recurrence relation procedure. We finally generalized this study to the edge dual of uncorrelated random networks. Although the critical temperature of Ising model on the edge dual network is higher than the one in the random network, both quantities become infinite in the same point, that is when the second moment of the degree distribution of random network $\langle k^2 \rangle$ becomes infinite. This fact reflects the robustness of edge-dual networks against thermal fluctuations, a property which can be attributed to the large number of triangles and the special structure of the edge-dual networks.

We also derived the critical behavior of Ising model on edge-dual network of an uncorrelated random scale free network. The results show that this behavior is significantly different from the one seen in the uncorrelated random networks. Moreover, we gave a simple relation between the partition function of an Ising model with nearest- and next-nearest-neighbor interactions on a tree like network and its edge dual. High- and low-temperature expansions of the partition

function of Ising model on the edge dual of random networks were also derived.

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